

Quasi-Chemical Approximation for Nonrandomness in the Hole Theory of Polymeric Fluids. 2. Miscibility Behavior of Binary Systems: Polystyrene in Cyclohexane or Methyl Acetate

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ABSTRACT: The nonrandom mixing version of the Holey Huggins theory for pure components presented in a previous paper is extended to binary mixtures and is applied to the miscibility behavior of the systems polystyrene/methyl acetate and polystyrene/cyclohexane. Comparison with the experimental data shows that the present theory is successful in predicting the miscibility behavior and its pressure dependence for these two systems. For polystyrene/cyclohexane the results of the random mixing and nonrandom mixing theory are comparable. For this system one indeed does not expect that nonrandomness will be very important. For the system polystyrene/methyl acetate the agreement between theory and experiment improves by applying the nonrandom mixing theory, indicating that nonrandom effects are of importance.

I. Introduction

In a previous paper¹ the effects of nonrandom mixing of polymer segments with holes in the Holey Huggins (HH) theory^{2,3} was accounted for by application of the quasi-chemical approximation of Guggenheim.⁴ The purpose of this paper is to report on (1) a generalization of the theory to polymer blends and polymer solutions and (2) the application to the miscibility behavior of binary systems. The theory is compared to experimental miscibility data at atmospheric and elevated pressure for solutions of polystyrene in methyl acetate and polystyrene in cyclohexane. The results show that the present theory is as good as or, in some respects, better than the original HH and SS theories.

The present theory is regarded as an intermediate but necessary step toward the establishment of an equation of state theory involving polymer systems with, e.g., strong directional interactions. In these systems, considerable deviation from complete randomness may be expected, and this can be handled in a similar fashion as discussed presently. So far, the present theory has already been extended to random copolymer systems and homopolymer/random copolymer blends, which will be presented in a forthcoming paper. The development of an equation of state theory for self-associated polymer systems in which very strong specific interactions play an important role is in progress.

II. Theory

(a) Model System. Consider a binary mixture of monodisperse components A and B. The following parameters characterize the system: N_A (N_B), the number of molecules of component A (B); s_A (s_B), the number of segments of each molecule A (B); $3c_A$ ($3c_B$), the number of external degrees of freedom of a molecule A (B); M_{0A} (M_{0B}), the molar mass of component A (B); P , V , and T , the pressure, total volume, and temperature of the system. For the sake of convenience, we introduce some other parameters as given below.

N the total number of molecules
 n the total number of moles of the mixture

N_h the number of holes on the lattice
 z the coordination number of the lattice
 y the occupied site fraction, $y = (N_A s_A + N_B s_B) / (N_A s_A + N_B s_B + N_h)$
 ϕ_A the segment fraction of component A (B), $\phi_A = (N_A s_A) / (N_A s_A + N_B s_B) = 1 - \phi_B$
 c_{sA} the flexibility parameter per segment, $c_{si} = c_i / s_i$ ($i = A, B$)
 zq_A the number of external contact sites of each molecule A (B), $zq_i = s_i(z - 2) + 2$ ($i = A, B$)
 $\langle s \rangle$ the average number of segments, $\langle s \rangle = (\phi_A / s_A + \phi_B / s_B)^{-1}$
 $\langle c_s \rangle$ the average flexibility parameter, $\langle c_s \rangle = \phi_A c_{sA} + \phi_B c_{sB}$
 $\langle M_0 \rangle$ the average molar mass, $\langle M_0 \rangle = \phi_A M_{0A} + \phi_B M_{0B}$
 u the contact site fraction of component A, $u = N_A z q_A / (N_A z q_A + N_B z q_B + z N_h)$
 v the contact site fraction of component B, $v = N_B z q_B / (N_A z q_A + N_B z q_B + z N_h)$
 w the contact site fraction of holes, $w = z N_h / (N_A z q_A + N_B z q_B + z N_h)$
 q the total contact site fraction of components A and B, $q = u + v$
 Q the total number of external contact pairs, $Q = (N_A z q_A + N_B z q_B + z N_h) / 2$

To take nonrandomness of the system into account, we need extra microscopic parameters to describe the number of contacts between different species. For a binary mixture, there are six different kinds of contacts; thus three extra parameters, denoted by X_1 , X_2 , and X_3 , are necessary. Their meanings are self-evident when one looks at the number of different kinds of contacts as cited in Table I.

The derivation of the theory is quite similar to that for pure components and is discussed in detail in the Appendix. Here only the main result of the derivation, i.e., the expression for the Helmholtz free energy of the system is presented.

(b) The Helmholtz Free Energy. According to the present theory, the free energy of the system is given by

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Table I
Enumeration of Contacts in Mixture

kind of contact pair	no. of contact pairs
segments a-b or b-a	QX_1
segment b-hole or hole-segment b	QX_2
segment a-hole or hole-segment a	QX_3
segments a-a	$Q(u - X_1 - X_3)$
segments b-b	$Q(v - X_1 - X_2)$
hole-hole	$Q(w - X_2 - X_3)$

$$A/n(s)RT = \phi_A \ln \phi_A/s_A + \phi_B \ln \phi_B/s_B +$$

$$\frac{1}{(s)} \ln y + \frac{1-y}{y} \ln(1-y) - \frac{1-\langle\alpha\rangle y}{\gamma y} \ln(1-\langle\alpha\rangle y) -$$

$$\frac{z(1-\langle\alpha\rangle)}{2q} [2u \ln u + 2v \ln v + 2w \ln w - (u - \bar{X}_1 -$$

$$\bar{X}_3) \ln(u - \bar{X}_1 - \bar{X}_3) - (v - \bar{X}_1 - \bar{X}_2) \ln(v - \bar{X}_1 - \bar{X}_2) -$$

$$(w - \bar{X}_2 - \bar{X}_3) \ln(w - \bar{X}_2 - \bar{X}_3) - 2\bar{X}_1 \ln \bar{X}_1 - 2\bar{X}_2 \ln \bar{X}_2 -$$

$$2\bar{X}_3 \ln \bar{X}_3] - \langle c_s \rangle \ln \left\{ \langle v^* \rangle \tilde{\omega} \left[1 - \beta \left(1 - \frac{\bar{X}}{q} \right) \right]^3 \right\} +$$

$$\frac{\langle c_s \rangle}{2T} \left(1 - \frac{\bar{X}}{q} \right) (A\tilde{\omega}^{-4} - 2B\tilde{\omega}^{-2}) + k_1 \langle c_s \rangle \ln(k_2 \langle M_0 \rangle T) \quad (1)$$

In eq 1, the first five terms on the right-hand side account for the combinatorial entropy of mixing randomly holes and segments of both components. The following terms in the square brackets are extra contributions due to nonrandom mixing, where \bar{X}_1 , \bar{X}_2 , and \bar{X}_3 are the values of X_1 , X_2 , and X_3 derived by the maximization condition in a similar manner as for pure component systems (see Appendix). The next two terms are contributions from the free volume and internal energy. The last term, derived from the kinetic part of the partition function, was originally introduced by Jain and Simha⁵ to account for the influence of mixing on the external degrees of freedom of the system. This term is also present in the HH theory, although in the relevant paper it has not been shown explicitly.³ The parameters k_1 and k_2 are numerical constants, and $\langle M_0 \rangle$ is the average molar mass. Starting from the Helmholtz free energy, all other thermodynamic properties, e.g., the equation of state and miscibility behavior, can be derived in a straightforward way.

III. Applications

(a) **Miscibility Behavior of the System Polystyrene/Methyl Acetate and Its Pressure Dependence.** Polystyrene (PS)/methyl acetate (MeOAc) is an interesting system which shows an upper critical miscibility temperature (UCMT) at low temperatures as well as a lower critical miscibility temperature (LCMT) at high temperatures.^{6,7} The relative location of the LCMT miscibility gap with respect to the UCMT miscibility gap ensures that the LCMT is governed by free volume parameters and therefore can be explained by hole theories. The miscibility behavior of this system according to the HH theory has been studied earlier.⁸ In that study, the pure component parameters are obtained from the experimental equation of state data of both components, and the intermolecular parameters are determined from a single experimental UCMT critical point.⁷ With these parameters, not only a satisfactory description of both UCMT and LCMT phase behavior is obtained but also the pressure dependences of UCMT and LCMT consolute states are predicted, at least qualitatively. Here, the same analysis is presented for the nonrandom mixing theory and the results are compared to those of the HH theory. The pure

Table II
Molecular Parameters for PS and MeOAc According to the Present Theory and the HH Theory^a

		ϵ^* (J/mol)	v^* (10^{-5} m ³ /mol)	c_s	M_0 (10^{-3} kg/mol)
PS	present	5471.4	7.7068	0.65438	81
	HH	5545.9	8.1539	0.68805	86
MeOAc	present	6083.6	6.8225	1.6501	74
	HH	5243.3	6.6813	1.5480	74

^a Experimental data taken from refs 13 and 14.

Table III
Intermolecular Parameters Determined from the UCMT Critical Point of PS($M=179$ kg/mol)/MeOAc^a

theory	δ_{es}^b	δ_{vs}^b
present	1.022 815 3	0.996 371 05
HH	1.029 971 5	0.990 110 00

^a Experimental data taken from ref 7. ^b By definition, $\epsilon_{ab}^* = \delta_{es}(\epsilon_{aa}^* \epsilon_{bb}^*)^{1/2}$ and $v_{ab}^* = \delta_{vs}[(v_{aa}^*)^{1/3} + (v_{bb}^*)^{1/3}]/2$.

component parameters and intermolecular parameters according to the present theory and the HH theory are summarized in Tables II and III, respectively.

As pointed out by Prigogine, disparity in segmental volume should be limited to approximately 10% to conform to the basic assumptions adopted in the lattice theory.⁹ If the segment of PS is taken equal to its molecular repeating unit, the resulting characteristic volumes of PS and MeOAc differ by more than 40%. To conform to the assumption underlying the lattice theory, the segmental molar mass of PS is chosen smaller than the molar mass of the chemical repeat unit. This somewhat arbitrary adjustment is probably an inherent drawback of all "on-lattice" theories. Fortunately, a change in the segmental molar mass does not alter the quality of the equation of state description. Only a change in all molecular parameters proportional to the change in the segmental molar mass is observed.⁸ However, changes in the segmental molar mass result in differences in the location of the predicted LCMT miscibility gap relative to the UCMT miscibility gap. For a given value of M_{0PS} (complying with the 10% rule) a set of cross parameters can be estimated from an exact fit of a given UCMT critical point. The location of the predicted LCMT critical point is different for each selected value of M_{0PS} .

The value of M_0 for PS shown in Table II offers a good description of the location of both UCMT and LCMT phase behavior. For example, the LCMT spinodal curves calculated with the molecular parameters listed in Tables II and III are shown in Figure 1, along with the experimental cloud points,⁶ which are taken to be equal to the binodal points considering the monodispersity of the polymer samples. It is obvious that the present theory gives a better description than the HH theory does. Furthermore, according to the HH theory, it is not possible to improve the location of the LCMT miscibility gaps by changing the value of M_{0PS} . However, if one notices that the intermolecular parameters used in calculating the LCMT spinodal curves were obtained from the UCMT critical point data, both the present theory and the HH theory should be regarded as satisfactory in describing the miscibility behavior of the system PS/MeOAc. It should not be surprising that the best value of M_0 for the present theory is slightly different from that for the HH theory.

Having obtained the complete set of molecular parameters which lead to a good description of the atmospheric pressure miscibility behavior, we are now in a position to predict other thermodynamic properties. Here, our pri-

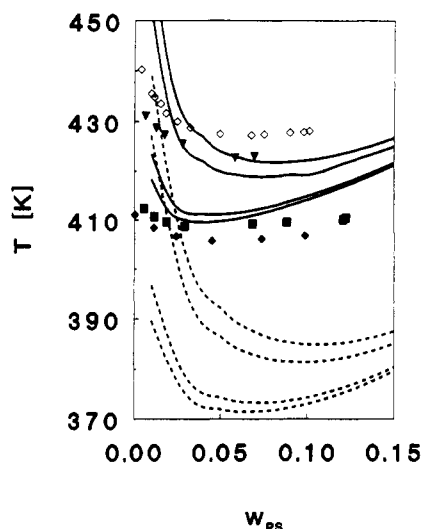


Figure 1. Experimental binodal points for four PS molar masses in MeOAc (M in kg/mol) (\diamond) $M = 48$; (∇) $M = 65$; (\blacksquare) $M = 177$; (\blacklozenge) $M = 246$,⁶ and the theoretical spinodal curves given by the present theory (—) and the HH theory (---) for the same molar masses. From top to bottom the molar mass increases.

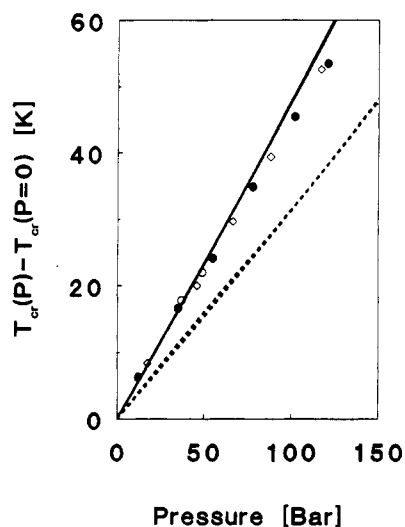


Figure 2. Experimental pressure dependence of UCM temperatures T_{cr} estimated from original cloud point data for three PS molar masses in MeOAc (M in kg/mol)¹⁰ (\diamond) $M = 97$; (\bullet) $M = 160$; (\circ) $M = 498$ and pressure dependence of the LCM temperature given by the present theory (—) and the HH theory (---). The data are presented as $T_{cr}(P) - T_{cr}(P=0)$.

mary interest is the pressure dependence of the critical points, since the HH theory has proven successful in this respect. The predicted and experimental pressure dependence of the LCMT and UCMT critical temperatures is shown in Figures 2 and 3. It may be seen that both theories can predict reasonably the pressure dependence of both LCMT and UCMT critical temperatures. In the case of the LCMT critical temperature, the prediction made by the present theory is almost quantitative. Also for the UCMT critical temperature, the present theory gives a better prediction of the pressure dependence, especially at higher molar masses. For the low molar masses, the pressure dependence is not reproduced exactly. Nevertheless the significant change in pressure dependence observed experimentally is also observed theoretically, although at a much lower molar mass. Thus the molar mass dependence is predicted qualitatively. In comparison, the predictions concerning the molar mass dependence according to the HH theory can be regarded to be less satisfactory.

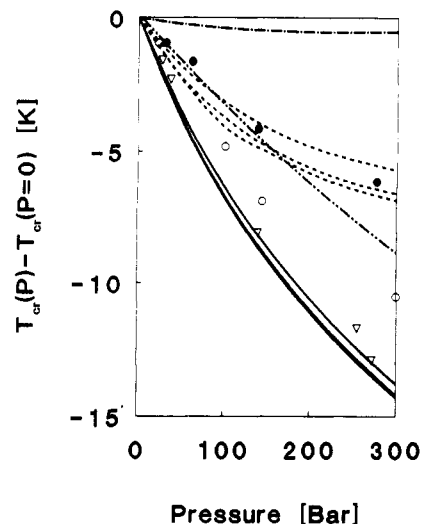


Figure 3. Experimental pressure dependence for UCM temperatures T_{cr} estimated from original cloud point data for three PS molar masses in MeOAc (M in kg/mol)¹⁰ (\bullet) $M = 160$; (\circ) $M = 498$; (∇) $M = 860$ and pressure dependence of the UCM temperature given by the present theory (—) and the HH theory (---). The data are presented as $T_{cr}(P) - T_{cr}(P=0)$. To illustrate the pressure sensitivity, computed results are shown for the system (chain length $s = 100$; HH (---) and NRMHH (—·—)).

Table IV
Molecular Parameters for PS and CH Obtained from PVT Data^{13,15}

	ϵ^* (J/mol)	v^* (10^{-5} m ³ /mol)	c_s	M_0 (10^{-3} kg/mol)
PS	7052.7	9.8754	0.87663	104
CH	7056.0	9.6681	1.7296	84.16

Table V
Intermolecular Parameters for PS/CH Estimated from the UCM Critical Condition of PS($M=520$ kg/mol)/CH¹¹

δ_{es}	δ_{vs}
0.985 657 74	1.007 225 0

(b) Pressure Dependence of the UCMT Critical Temperature of the System Polystyrene/Cyclohexane. As another example of the application of the present theory, we will discuss briefly the system PS/cyclohexane (CH). As mentioned before, the HH theory is very successful in predicting the subtle changes in the UCMT critical temperature with pressure of the system PS/CH.³ The same is true for the present theory. Here, we follow exactly the same procedure as in ref 3. The molecular parameters for the pure components obtained from equation of state data and the intermolecular parameters obtained from a single UCMT consolute point ($M_{PS} = 520$ kg/mol) are summarized in Tables IV and V. In this case, the segmental molar mass of PS can be taken equal to that of the chemical repeat unit, because the characteristic volumes of PS and CH are almost equal. The predicted pressure dependence of the UCMT critical temperature is shown in Figure 4 in comparison with the experimental data of Wolf and Geerissen.¹¹ At first inspection it appears that the nonrandom mixing HH (NRMHH) theory does not predict the UCMT pressure dependence as good as the random mixing theory. However, it should be noted that for the HH theory some extra adjustment has been done. To obtain spinodal curves in better agreement with experimental data, the number of external degrees of freedom for cyclohexane was increased to $c_{CH} = 1.8$.¹² For the NRMHH theory no such adjustment was introduced. Much better agreement between theory and experiment can be expected if some adjustment of the c parameter is

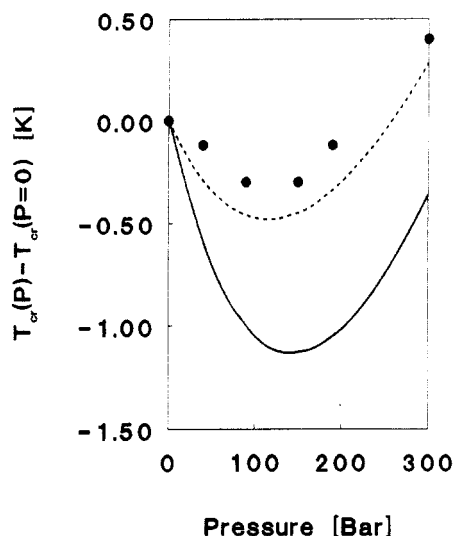


Figure 4. Pressure dependence of UCM temperature T_{cr} of PS- ($M=600$ kg/mol)/CH according to the present theory (—) and the HH (---) theory. Experimental data (●) presented as $T_{cr}(P) - T_{cr}(P=0)$ are taken from ref 11.

allowed for in the nonrandom mixing theory. However, this option will not be pursued here. Conversely, only qualitative agreement with experiment is obtained for the HH theory if c_{CH} is not adjusted. Therefore, it can be concluded that the overall agreement between theory and experiment is improved including nonrandom mixing effects.

IV. Conclusions

The hole theory for the disordered fluid state including nonrandom mixing has been successfully extended to strictly binary systems including homopolymer solutions and mixtures or blends. The comparison presented here was primarily intended to gauge the effects of quasi-chemical approximation on the description and prediction of thermodynamic properties, in particular phase behavior.

For the system PS/CH, which can be considered as a model system for apolar interactions, the NRMHH and HH theories are practically identical. From the equation of state behavior of the pure constituents nonnegligible nonrandom mixing effects are observed.¹ Considering solutions of polystyrene and cyclohexane, the deviations from nonrandomness are also present but appear to be relatively insensitive to composition, which is reflected in the minor influence of nonrandomness on thermodynamic properties related to compositional derivatives, e.g., spinodals, critical conditions, etc.

For the system PS/MeOAc relatively favorable interactions are suggested by the cross interaction parameters estimated from the HH theory. The large deviations from the geometric mean for the energetic parameter ϵ_{ab}^* and the hard-sphere average for the segmental volume v_{ab}^* suggest that considerable deviations from randomness can be expected. The NRMHH theory should be suited to deal with such deviations in a consistent manner. Indeed for this theory the cross parameters ϵ_{ab}^* and v_{ab}^* are much closer to the geometric and hard-sphere averages, and, more importantly, a better overall agreement between theory and experiment is obtained.

From these results it can be concluded that the quasi-chemical approximation effectively offers an improved evaluation of the partition function. More (direct) evidence concerning the accuracy of an analytical approximation to the exact partition function for a particular model can be obtained from a comparison with Monte Carlo

simulation results. Presently, the hole model has not been the subject of such a study. However, for the simple lattice-fluid model in which cell contributions are neglected a comparison with MC results is in progress. The preliminary results show that the quasi-chemical approach presents a very accurate approximation to the exact partition function at densities relevant for pure polymers and mixtures. This is reflected in a successful prediction of, e.g., equation of state behavior and the number and type of segmental contacts. A more detailed comparison will be the subject of a forthcoming paper.

Finally, we remark once more that the present theory is regarded as a starting point for a theory of associating systems. The ultimate aim there is the establishment of connections between thermodynamic equilibrium constants and the equation of state.

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Appendix: Derivation of the Theory for Binary Mixtures

(1) Partition Function. The partition function of the system is given by

$$Z(N, \phi_A, y, V, T) = \sum_{X_1, X_2, X_3} Z_x(N, \phi_A, y, V, T, X_1, X_2, X_3) \quad (\text{A.1})$$

where the summation runs over the three X parameters. The partition function of the subsystem Z_X can be factorized:

$$Z_X = g l_f^{3\langle c_s \rangle N \langle s \rangle} \exp(-E_0/kT) \quad (\text{A.2})$$

where the parameters c_s and s in the pure component theory have been replaced by their average values $\langle c_s \rangle$ and $\langle s \rangle$. According to Guggenheim,⁴ the combinatorial factor g reads

$$g = \frac{[Q(u - X_1^* - X_3^*)]! [Q(v - X_1^* - X_2^*)]! [Q(w - X_2^* - X_3^*)]!}{[Q(u - X_1 - X_3)]! [Q(v - X_1 - X_2)]! [Q(w - X_2 - X_3)]!} \times \frac{[(QX_1^*)!]^2 [(QX_2^*)!]^2 [(QX_3^*)!]^2}{[(QX_1)!]^2 [(QX_2)!]^2 [(QX_3)!]^2} \quad (\text{A.3})$$

where X_i^* ($i = 1, 2, 3$) are the corresponding values of X_i in case of random mixing. The expression for the segmental free length takes almost the same form as for pure components, if only we define X to be the sum of X_2 and X_3

$$l_f = \left(1 - \frac{X_2 + X_3}{u + v}\right) \langle l_s \rangle + \frac{X_2 + X_3}{u + v} \langle l_g \rangle = \left(1 - \frac{X}{q}\right) \langle l_s \rangle + \frac{X}{q} \langle l_g \rangle \quad (\text{A.4})$$

Also, the definitions of the solid-like and gas-like free lengths $\langle l_s \rangle$ and $\langle l_g \rangle$ are similar to those in the pure component theory, viz.

$$\langle l_s \rangle = (\tilde{\omega}^{1/3} - 2^{-1/6}) \langle v^* \rangle^{1/3} \equiv \langle \tilde{l}_s \rangle \langle v^* \rangle^{1/3} \quad (\text{A.5})$$

$$\langle l_g \rangle = \tilde{\omega}^{1/3} \langle v^* \rangle^{1/3} \equiv \langle \tilde{l}_g \rangle \langle v^* \rangle^{1/3} \quad (\text{A.6})$$

where $\langle v^* \rangle$ is the average hard-core volume, which will be defined later, $\tilde{\omega} = yv/\langle v^* \rangle$ is the reduced cell volume, and $v = V/N\langle s \rangle$ is the volume per segment. The internal energy

E_0 now becomes

$$E_0 = Q(u - X_1 - X_3)\epsilon_{aa} + Q(v - X_1 - X_2)\epsilon_{bb} + 2QX_1\epsilon_{ab} \quad (\text{A.7})$$

or

$$E_0 = Qu\epsilon_{aa} + Qv\epsilon_{bb} + Q(2\epsilon_{ab} - \epsilon_{aa} - \epsilon_{bb})X_1 - Q\epsilon_{bb}X_2 - Q\epsilon_{aa}X_3 \quad (\text{A.8})$$

If we define

$$\epsilon_1 = 2\epsilon_{ab} - \epsilon_{aa} - \epsilon_{bb} \quad (\text{A.9})$$

$$\epsilon_2 = -\epsilon_{bb} \quad (\text{A.10})$$

$$\epsilon_3 = -\epsilon_{aa} \quad (\text{A.11})$$

then the internal energy can be expressed as

$$E_0 = Qu\epsilon_{aa} + Qv\epsilon_{bb} + Q(\epsilon_1X_1 + \epsilon_2X_2 + \epsilon_3X_3) \quad (\text{A.12})$$

In the above equations, ϵ_{aa} , ϵ_{bb} , and ϵ_{ab} obey Lennard-Jones 6-12 potentials

$$\epsilon_{aa} = \epsilon_{aa}^*(A\tilde{\omega}_a^{-4} - 2B\tilde{\omega}_a^{-2}) \quad (\text{A.13})$$

$$\epsilon_{bb} = \epsilon_{bb}^*(A\tilde{\omega}_b^{-4} - 2B\tilde{\omega}_b^{-2}) \quad (\text{A.14})$$

$$\epsilon_{ab} = \epsilon_{ab}^*(A\tilde{\omega}_{ab}^{-4} - 2B\tilde{\omega}_{ab}^{-2}) \quad (\text{A.15})$$

The different indices in $\tilde{\omega}_i$ ($i = a, b, ab$) in the above equations indicate that these volumes are reduced with respect to different kinds of hard-core volumes.

(2) **Maximization of Z_x .** If we denote the values of X_i ($i = 1, 2, 3$) which make Z_X maximum by \bar{X}_i , we have

$$Z(N, \phi_A, \gamma, V, T) \approx Z_x(N, \phi_A, \gamma, V, T, \bar{X}_1, \bar{X}_2, \bar{X}_3) \quad (\text{A.16})$$

where \bar{X}_i ($i = 1, 3$) are determined by

$$\left. \frac{\partial \ln Z_x}{\partial X_i} \right|_{N, \phi_A, \gamma, V, T, X_j \neq i} = 0 \quad (\text{A.17})$$

The logarithm of Z_X reads

$$\begin{aligned} \ln Z_X = & \ln [Q(u - X_1 - X_3)!] + \ln [Q(v - X_1 - X_2)!] + \\ & \ln [Q(w - X_2 - X_3)!] + 2 \ln (QX_1)! + 2 \ln (X_2)! + \\ & 2 \ln (X_3)! - 3\langle c_s \rangle N \langle s \rangle \ln \left[1 - \beta \left(1 - \frac{X}{q} \right) \right] + \frac{Q}{kT} (\omega_1 X_1 + \\ & \omega_2 X_2 + \omega_3 X_3) + \text{terms independent of } X \end{aligned} \quad (\text{A.18})$$

where $\beta = 2^{-1/6}\tilde{\omega}^{-1/3}$. Note that β (or, more exactly, $\langle v^* \rangle$) is a function of X_1 , X_2 , and X_3 as may be seen later. Combination of eqs A.17 and A.18 yields the following set of equations which determine the values of \bar{X}_1 , \bar{X}_2 , and \bar{X}_3 :

$$(\bar{X}_1\eta_{e1})^2 = (u - \bar{X}_1 - \bar{X}_3)(v - \bar{X}_1 - \bar{X}_2)\eta_{s1}$$

$$(\bar{X}_2\eta_{e2})^2 = (v - \bar{X}_1 - \bar{X}_2)(w - \bar{X}_2 - \bar{X}_3)\eta_{s2}$$

$$(\bar{X}_3\eta_{e3})^2 = (u - \bar{X}_1 - \bar{X}_3)(w - \bar{X}_2 - \bar{X}_3)\eta_{s3} \quad (\text{A.19})$$

where η_{ei} ($i = 1, 2, 3$) are given by $\eta_{ei} = \exp(\epsilon_i/2kT)$ and η_{s1} through η_{s3} are defined as

$$\eta_{s1} = \exp \left[- \frac{2\langle c_s \rangle q}{z(1 - \langle \alpha \rangle)} \frac{\beta \left(1 - \frac{\bar{X}}{q} \right)}{1 - \beta \left(1 - \frac{\bar{X}}{q} \right)} \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_1} \right] \quad (\text{A.20})$$

$$\eta_{s2} = \exp \left[\frac{6\langle c_s \rangle}{z(1 - \langle \alpha \rangle)} \frac{2^{-1/6}}{l_f} \right] \times \exp \left[- \frac{2\langle c_s \rangle q}{z(1 - \langle \alpha \rangle)} \frac{\beta \left(1 - \frac{\bar{X}}{q} \right)}{1 - \beta \left(1 - \frac{\bar{X}}{q} \right)} \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_2} \right] \quad (\text{A.21})$$

$$\eta_{s3} = \exp \left[\frac{6\langle c_s \rangle}{z(1 - \langle \alpha \rangle)} \frac{2^{-1/6}}{l_f} \right] \times \exp \left[- \frac{2\langle c_s \rangle q}{z(1 - \langle \alpha \rangle)} \frac{\beta \left(1 - \frac{\bar{X}}{q} \right)}{1 - \beta \left(1 - \frac{\bar{X}}{q} \right)} \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_3} \right] \quad (\text{A.22})$$

where $\langle \alpha \rangle = \gamma(1 - 1/\langle s \rangle)$, $\gamma = 2/z$, and $l_f = l_t/\langle v^* \rangle^{1/3}$ is the reduced free length. The partial derivatives $\partial \langle v^* \rangle / \partial \bar{X}_i$ in the above equations may be calculated from the definition of $\langle v^* \rangle$ (cf. eq A.35).

(3) **Determination of X_i^* .** Following Guggenheim,⁴ X_1^* , X_2^* , and X_3^* are determined from the minimization condition

$$\partial Z_X^{(0)} / \partial X_i = 0 \quad (i = 1, 2, 3) \quad (\text{A.23})$$

where

$$\begin{aligned} \ln Z_X^{(0)} = & \ln [Q(u - X_1 - X_3)!] + \ln [Q(v - X_1 - X_2)!] + \\ & \ln [Q(w - X_2 - X_3)!] + 2 \ln (QX_1)! + 2 \ln (QX_2)! + \\ & 2 \ln (QX_3)! + \text{constant} \end{aligned} \quad (\text{A.24})$$

The resulting equations for the X_i^* 's are

$$X_1^{*2} = (u - X_1^* - X_3^*)(v - X_1^* - X_2^*)$$

$$X_2^{*2} = (v - X_1^* - X_2^*)(w - X_2^* - X_3^*)$$

$$X_3^{*2} = (u - X_1^* - X_3^*)(w - X_2^* - X_3^*) \quad (\text{A.25})$$

or equivalently

$$[w - (u + v)]X_1^* + [(v + w) - u]X_2^* = v(w - u)$$

$$[w - (u + v)]X_1^* + [(u + w) - v]X_3^* = u(w - v)$$

$$[u - (v + w)]X_2^* + [(u + w) - v]X_3^* = w(u - v) \quad (\text{A.26})$$

From the above set of linear equations we obtain the following expressions for X_2^* and X_3^* in terms of X_1^*

$$\begin{aligned} X_2^* &= \frac{v(w - u) + [(u + v) - w]X_1^*}{(v + w) - u} \\ X_3^* &= \frac{u(w - v) + [(u + v) - w]X_1^*}{(u + w) - v} \end{aligned} \quad (\text{A.27})$$

Notice that, since the determinant of the set of linear equations (eq A.26) equals zero, there are an infinite number of solutions for X_1^* . However, if we consider the total fraction of segment-hole contacts X^* ($X^* = X_2^* + X_3^*$) and the total contact fraction of polymer segments q , X_1^* may be determined by the requirement that the relationship between X^* and q should be the same as that for the pure component. In other words, when we consider the polymer as a whole entity, we should have $X^* = q(1 - q)$, as in the pure component theory. At present, $q = u$

+ v and $1 - q = w$, so we have

$$X^* = \frac{w^2(u+v) - w(u^2 + v^2) + 2w(u+v-w)X_1^*}{(v+w-u)(u+w-v)} = (u+v)w \quad (\text{A.28})$$

From eq A.28, we obtain $X_1^* = uv$ and, thereafter, $X_2^* = uv$ and $X_3^* = uv$.

(4) Appropriate Partition Function. Combination of eqs A.16 and A.2 gives the configurational partition function of the system

$$Z_{\text{conf}} = g(N, \phi_A, y, \bar{X}_1, \bar{X}_2, \bar{X}_3) [l_f(\phi_A, y, V, \bar{X}_1, \bar{X}_2, \bar{X}_3)]^{3\langle c_s \rangle N(s)} \times \exp[-E_0(N, \phi_A, y, \bar{X}_1, \bar{X}_2, \bar{X}_3)/kT] \quad (\text{A.29})$$

The combinatorial factor g has the form

$$g(N, \phi_A, y, \bar{X}_1, \bar{X}_2, \bar{X}_3) = N_{\text{hh}} \frac{[Q(u - X_1^* - X_3^*)][Q(v - X_1^* - X_2^*)][Q(w - X_2^* - X_3^*)]}{[Q(u - \bar{X}_1 - \bar{X}_3)][Q(v - \bar{X}_1 - \bar{X}_2)][Q(w - \bar{X}_2 - \bar{X}_3)]} \times \frac{[(QX_1^*)!]^2[(QX_2^*)!]^2[(QX_3^*)!]^2}{[(Q\bar{X}_1)!]^2[(Q\bar{X}_2)!]^2[(Q\bar{X}_3)!]^2} \quad (\text{A.30})$$

According to HH theory,³ we have

$$\ln N_{\text{hh}} = N(s) \left[-\phi_A \ln \phi_A/s_A - \phi_B \ln \phi_B/s_B - \frac{1}{\langle s \rangle} \ln y - \frac{1-y}{y} \ln(1-y) + \frac{1}{\gamma} \left(\frac{1-\langle \alpha \rangle y}{y} \right) \ln(1-\langle \alpha \rangle y) \right] \quad (\text{A.31})$$

and the free length per segment is

$$l_f(\phi_A, y, V, \bar{X}_1, \bar{X}_2, \bar{X}_3) = \tilde{\omega}^{1/3} \left[1 - \beta \left(1 - \frac{\bar{X}}{q} \right) \right] \langle v^* \rangle^{1/3} \quad (\text{A.32})$$

The expression for the internal energy reads

$$-E_0(N, \phi_A, y, V, \bar{X}_1, \bar{X}_2, \bar{X}_3)/kT = -\frac{N\langle s \rangle z(1-\langle \alpha \rangle)}{2qRT} [(u - \bar{X}_1 - \bar{X}_3)\epsilon_{aa}^*(A\tilde{\omega}_{aa}^{-4} - 2B\tilde{\omega}_{aa}^{-2}) + (v - \bar{X}_1 - \bar{X}_2)\epsilon_{bb}^*(A\tilde{\omega}_{bb}^{-4} - 2B\tilde{\omega}_{bb}^{-2}) + 2\bar{X}_1\epsilon_{ab}^*(A\tilde{\omega}_{ab}^{-4} - 2B\tilde{\omega}_{ab}^{-2})] \quad (\text{A.33})$$

Comparing eq A.33 with the corresponding formulas for the pure component

$$-E_0/kT = -\frac{Nsz(1-\alpha)}{2qRT} (q - \bar{X})\epsilon^*(A\tilde{\omega}^{-4} - 2B\tilde{\omega}^{-2}) \quad (\text{A.34})$$

we obtain the mixing rules:

$$(u - \bar{X}_1 - \bar{X}_3)\epsilon_{bb}^*v_{aa}^{*4} + (v - \bar{X}_1 - \bar{X}_2)\epsilon_{bb}^*v_{bb}^{*4} + 2\bar{X}_1\epsilon_{ab}^*v_{ab}^{*4} = (q - \bar{X})\langle \epsilon^* \rangle \langle v^* \rangle^4$$

$$(u - \bar{X}_1 - \bar{X}_3)\epsilon_{aa}^*v_{aa}^{*2} + (v - \bar{X}_1 - \bar{X}_2)\epsilon_{bb}^*v_{bb}^{*2} + 2\bar{X}_1\epsilon_{ab}^*v_{ab}^{*2} = (q - \bar{X})\langle \epsilon^* \rangle \langle v^* \rangle^2 \quad (\text{A.35})$$

where ϵ_{aa}^* (ϵ_{bb}^*) and v_{aa}^* (v_{bb}^*) are related to the interaction between segments of identical molecules, ϵ_{ab}^* and v_{ab}^* are cross interaction parameters related to the interaction between A-B segments, and $\bar{X} = \bar{X}_2 + \bar{X}_3$. Note that if

we let $\bar{X}_i = X_i^*$ ($i = 1, 2, 3$) eq A.35 recovers the mixing rules in the HH theory. Using $\langle \epsilon^* \rangle$ and $\langle v^* \rangle$ defined by eq A.35, the internal energy of the system may be written simply as

$$-E_0/kT = -N\langle s \rangle \frac{\langle c_s \rangle}{2\tilde{T}} \left[\left(1 - \frac{\bar{X}}{q} \right) (A\tilde{\omega}^{-4} - 2B\tilde{\omega}^{-2}) \right] \quad (\text{A.36})$$

where $\tilde{T} = T/T^*$ is the reduced temperature and

$$T^* = z(1 - \langle \alpha \rangle) \langle \epsilon^* \rangle / (\langle c_s \rangle R) \quad (\text{A.37})$$

Finally, the logarithm of the partition function of the system is given by

$$\ln Z_{\text{conf}}/N\langle s \rangle = -\phi_A \ln \phi_A/s_A - \phi_B \ln \phi_B/s_B - \frac{1}{\langle s \rangle} \ln y - \frac{1-y}{y} \ln(1-y) + \frac{1-\langle \alpha \rangle y}{\gamma y} \ln(1-\langle \alpha \rangle y) + \frac{z(1-\langle \alpha \rangle)}{2q} [2u \ln u + 2v \ln v + 2w \ln w - (u - \bar{X}_1 - \bar{X}_3) \ln(u - \bar{X}_1 - \bar{X}_3) - (v - \bar{X}_1 - \bar{X}_2) \ln(v - \bar{X}_1 - \bar{X}_2) - (w - \bar{X}_2 - \bar{X}_3) \ln(w - \bar{X}_2 - \bar{X}_3) - 2\bar{X}_1 \ln \bar{X}_1 - 2\bar{X}_2 \ln \bar{X}_2 - 2\bar{X}_3 \ln \bar{X}_3] + \langle c_s \rangle \ln \langle v^* \rangle + \langle c_s \rangle \ln \tilde{\omega} + 3\langle c_s \rangle \ln \left[1 - \beta \left(1 - \frac{\bar{X}}{q} \right) \right] - \frac{\langle c_s \rangle}{2\tilde{T}} \left(1 - \frac{\bar{X}}{q} \right) (A\tilde{\omega}^{-4} - 2B\tilde{\omega}^{-2}) \quad (\text{A.38})$$

(5) Helmholtz Free Energy and Equation of State. The free energy of the system is given by

$$A/n\langle s \rangle RT = \phi_A \ln \phi_A/s_A + \phi_B \ln \phi_B/s_B + \frac{1}{\langle s \rangle} \ln y + \frac{1-y}{y} \ln(1-y) - \frac{(1-\langle \alpha \rangle y)}{\gamma y} \ln(1-\langle \alpha \rangle y) - \frac{z(1-\langle \alpha \rangle)}{2q} [2u \ln u + 2v \ln v + 2w \ln w - (u - \bar{X}_1 - \bar{X}_3) \ln(u - \bar{X}_1 - \bar{X}_3) - (v - \bar{X}_1 - \bar{X}_2) \ln(v - \bar{X}_1 - \bar{X}_2) - (w - \bar{X}_2 - \bar{X}_3) \ln(w - \bar{X}_2 - \bar{X}_3) - 2\bar{X}_1 \ln \bar{X}_1 - 2\bar{X}_2 \ln \bar{X}_2 - 2\bar{X}_3 \ln \bar{X}_3] - \langle c_s \rangle \ln \left\{ \tilde{\omega} \langle v^* \rangle \left[1 - \beta \left(1 - \frac{\bar{X}}{q} \right) \right]^3 \right\} + \frac{\langle c_s \rangle}{2\tilde{T}} \left(1 - \frac{\bar{X}}{q} \right) (A\tilde{\omega}^{-4} - 2B\tilde{\omega}^{-2}) + k_1 \langle c_s \rangle \ln(k_2(M_0)T) \quad (\text{A.39})$$

where the last term on the right-hand side of the equation comes from the kinetic part of the partition function.⁵ The equilibrium value of the occupied site fraction y is determined by the minimization condition

$$\frac{\partial A}{\partial y}|_{N,V,T,\phi_A} = 0 \quad (\text{A.40})$$

and the equation of state of the binary system is given by

$$P = -\frac{\partial A}{\partial V}|_{N,T,y,\phi_A} \quad (\text{A.41})$$

Formally, eqs A.40 and A.41 are quite similar to the corresponding equations in the pure component theory. However, in the present case, the partial derivatives are much more complicated. Especially, one must be cautious about the reduced parameters, since now $\langle v^* \rangle$ and $\langle \epsilon^* \rangle$ are functions of \bar{X}_i ($i = 1, 2, 3$) and y , whereas the \bar{X}_i 's themselves are functions of y and v .

By straightforward algebra, one obtains the equation of state of the system in its reduced form

$$\frac{\bar{P}\bar{v}}{\bar{T}} = \frac{1}{1 - \beta\left(1 - \frac{\bar{X}}{q}\right)} + \frac{2}{\bar{T}}\left(1 - \frac{\bar{X}}{q}\right)(A\bar{\omega}^{-4} - B\bar{\omega}^{-2}) + \left(m_{21} + m_{31} + \frac{m_{41}}{\bar{T}}\right)\bar{v}\frac{\partial \bar{X}_1}{\partial \bar{v}} + \left(m_{22} + m_{32} + \frac{m_{42}}{\bar{T}}\right)\bar{v}\frac{\partial \bar{X}_2}{\partial \bar{v}} + \left(m_{23} + m_{33} + \frac{m_{43}}{\bar{T}}\right)\bar{v}\frac{\partial \bar{X}_3}{\partial \bar{v}} \quad (\text{A.42})$$

where

$$m_{21} = -\frac{z(1 - \langle \alpha \rangle)}{2q\langle c_s \rangle} \ln \left[\frac{(\bar{X}_1)^2}{(u - \bar{X}_1 - \bar{X}_3)(v - \bar{X}_1 - \bar{X}_2)} \right] \quad (\text{A.43})$$

$$m_{22} = -\frac{z(1 - \langle \alpha \rangle)}{2q\langle c_s \rangle} \ln \left[\frac{(\bar{X}_2)^2}{(v - \bar{X}_1 - \bar{X}_2)(w - \bar{X}_2 - \bar{X}_3)} \right] \quad (\text{A.44})$$

$$m_{23} = -\frac{z(1 - \langle \alpha \rangle)}{2q\langle c_s \rangle} \ln \left[\frac{(\bar{X}_3)^2}{(u - \bar{X}_1 - \bar{X}_3)(w - \bar{X}_2 - \bar{X}_3)} \right] \quad (\text{A.45})$$

$$m_{31} = -\frac{1}{1 - \beta\left(1 - \frac{\bar{X}}{q}\right)} \left[\beta\left(1 - \frac{\bar{X}}{q}\right) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_1} \right] \quad (\text{A.46})$$

$$m_{32} = \frac{1}{1 - \beta\left(1 - \frac{\bar{X}}{q}\right)} \left[\frac{3\beta}{q} - \beta\left(1 - \frac{\bar{X}}{q}\right) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_2} \right] \quad (\text{A.47})$$

$$m_{33} = \frac{1}{1 - \beta\left(1 - \frac{\bar{X}}{q}\right)} \left[\frac{3\beta}{q} - \beta\left(1 - \frac{\bar{X}}{q}\right) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_3} \right] \quad (\text{A.48})$$

$$m_{41} = -\frac{1}{2}\left(1 - \frac{\bar{X}}{q}\right) \left[(A\bar{\omega}^{-4} - 2B\bar{\omega}^{-2}) \frac{\partial \ln \langle \epsilon^* \rangle}{\partial \bar{X}_1} + 4(A\bar{\omega}^{-4} - B\bar{\omega}^{-2}) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_1} \right] \quad (\text{A.49})$$

$$m_{42} = -\frac{1}{2}\left[\left(1 - \frac{\bar{X}}{q}\right) \frac{\partial \ln \langle \epsilon^* \rangle}{\partial \bar{X}_2} - \frac{1}{q}\right] (A\bar{\omega}^{-4} - 2B\bar{\omega}^{-2}) - 2\left(1 - \frac{\bar{X}}{q}\right) (A\bar{\omega}^{-4} - B\bar{\omega}^{-2}) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_2} \quad (\text{A.50})$$

$$m_{43} = -\frac{1}{2}\left[\left(1 - \frac{\bar{X}}{q}\right) \frac{\partial \langle \epsilon^* \rangle}{\partial \bar{X}_3} - \frac{1}{q}\right] (A\bar{\omega}^{-4} - 2B\bar{\omega}^{-2}) - 2\left(1 - \frac{\bar{X}}{q}\right) (A\bar{\omega}^{-4} - B\bar{\omega}^{-2}) \frac{\partial \ln \langle v^* \rangle}{\partial \bar{X}_3} \quad (\text{A.51})$$

The partial derivatives in the above equations may be obtained either analytically or numerically. The way to proceed analytically has been explained in the previous publication.¹ In the present applications these derivatives were obtained numerically with the aid from a computer algorithm to determine derivatives of functions.¹⁶ The derivatives $\partial \bar{X}_i / \partial \bar{v}$, $\partial \langle \epsilon^* \rangle / \partial \bar{X}_i$, and $\partial \langle v^* \rangle / \partial \bar{X}_i$ for $i = 1-3$ are obtained from eqs A.19 and A.35, respectively, keeping the relevant parameters constant.

Notice that although the above theory is derived for a binary mixture of homopolymers, it is also valid for polymer solutions.

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